Technical Notes

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Application of the Finite-Element Method to Transonic Flows with Shocks

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Introduction

THE strength and location of weak shocks are of practical interest in aeronautical applications. The mesh refinement studies of the supersonic pocket over a circular cylinder using Chen's[1] finite-element program were carried out by Wadia,[2,3] and Shen.[4] Their studies indicate that the subsonic program of Chen[1] does not immediately lose its numerical stability at the critical Mach number but is capable of predicting a well-defined embedded supersonic pocket. However, Wadia's[3] study suggests that if the finite-element mesh contains an overwhelming number of supersonic nodes, the overall stability of the numerical scheme is adversely affected and the iterations do not converge. Shen[4] has remarked that the predicted shockless supersonic pocket at best represents a highly smeared approximation of the transonic solution which contains actual weak shocks, and the size of the supersonic pocket could be quite wrong.

For a weak shock to be a part of the final solution in transonic flow, the governing differential equation must be applied only in the domain from which the shock has been excluded by imaginary boundaries. The proper jump condition must be imposed across the shock boundary. There is no existing finite-element program, to our knowledge, developed along this line, which is tantamount to "shock-fitting." The finite-difference solution of the nonlinear-mixed-type equations has been studied by von Neumann and Richtmyer,[5] Lax-Wendroff,[6] MacCormack,[7] Murman and Cole,[8] and Jameson,[9] among others. Some of the successful finite-difference methods feature type-dependent difference formulas which turn out to have dissipative properties equivalent to the addition of artificial viscosity (or heat conduction) terms to the original equation, thereby capable of "shock-capturing" without the special consideration of discontinuous solutions. The captured shock manifests itself with a "thickness" of several mesh spacings. The advantage of the finite-element method diappears if it becomes necessary to depend on small mesh sizes comparable to those of the finite-difference method. In fact, the following results of our investigation suggest that "shock-fitting" might not be unthinkable for finite-element methods. The primary aim of this study is to demonstrate a new application of the well established and currently popular finite-element method.

Formulation and Numerical Analysis

We choose to study the simple one-dimensional Laval nozzle with shock as the model problem. By assuming transonic small perturbations, the governing equation is of the form (see Ref. 2)

$$\frac{d(V^2)}{d\xi} - \xi(1-V) = 0 \tag{1}$$

$$V = -\frac{2\gamma}{\gamma+1}(M^2-1), \quad \xi = 4\gamma\sqrt{\frac{\alpha}{\gamma+1}}\,x, \tag{1a}$$

$$\alpha = \frac{\text{throat height}}{2(\text{radius of curvature at the origin})}$$

where V is proportional to the perturbation from the sonic condition, $V>0$ for subsonic flow, and the x axis runs downstream with the origin at the throat, γ is the ratio of the specific heats, and M is the Mach number. The solution of interest satisfies the boundary conditions,

$$V(-a) = V_1 > 0 \tag{2a}$$

$$V(a) = V_2 > 0 \tag{2b}$$

plus the transonic jump condition at the shock location (see Ref. 4)

$$[V^2] = 0 \tag{2c}$$

The exact solution in closed form can be obtained easily. Both the continuous and the discontinuous solutions are sketched in Fig. 1. The saddle-point nature of the solutions near the sonic throat is well known, and it follows that several neighboring solutions prevail in the throat region that satisfy essentially the same pair of boundary values V_1 and V_2. If the pair defines an exact solution with a compression shock, with very small changes in V_1 and V_2, the exact solution becomes either one of the continuous solutions or another continuous solution with a rarefaction shock. Hence, in the numerical computation with the given pair of end conditions, the task of distinguishing the correct one among the competitive neighboring solutions becomes a delicate matter.

As a numerical experiment, the Galerkin method with a Fourier series approximation for V was studied by Wadia,[2] and the resulting set of quadratic equations was solved by Newton's method. Since the exact solution with shock was known, the exact Fourier coefficients were determined and used to start the iteration in the finite approximation. The iterated solution (up to 12 terms in the Fourier series expansion) always moved from the initial approximation and settled near one of the continuous solutions (for details see Refs. 2 and 4).

To carry to still higher-order approximations, it is convenient to resort to the finite-element representation with linear interpolation along the line, since the necessary quadratures become much simpler, and further instead of the Galerkin we adopt the slightly more complicated least squares. The segment $-a \le \xi \le a$ is subdivided into a finite number of unequal intervals. Using a linear variation of V in each element, the least-squares finite-element formulation

$$\frac{\partial}{\partial V_i}\int_{\xi_{i-1}}^{\xi_i} R^2 d\xi + \frac{\partial}{\partial V_i}\int_{\xi_i}^{\xi_{i+1}} R^2 d\xi = 0 \tag{3a}$$

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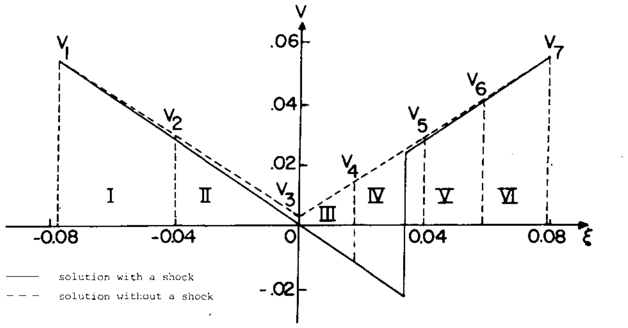

Fig. 1 Plot of V vs ξ: —— solution with a shock——— without a shock.

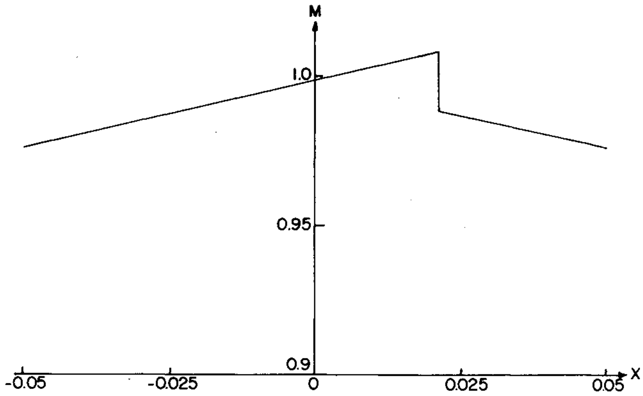

Fig. 2 Distribution of the Mach number along the axis of a one-dimensional Laval nozzle, $\alpha = 0.1$.

where

$$R = \frac{\mathrm{d}(V^2)}{\mathrm{d}\xi} - \xi(1 - V) \tag{3b}$$

$(\;)_i$ = value of the quantity at the ith node, $i = 1, 2, \ldots, n+1$, results in a set of $n-1$ cubic equations and the coupling of the equations is only with the nearest neighbors. Again, the solution is obtained by Newton's method. The exact values of V at the nodes are used again as the initial approximation, but with a moderate n, the successive iterations show the same tendency to move away and settle near the continuous solution, with an apparent supersonic pocket.

As the least-squares prcedure corresponds to an extremum variational principle, as $n \to \infty$ there can be doubt about reproducing the exact solution. The failure to do so at finite n is therefore due to the discretization error. Also, as the exact solution is quite close to two straight line segments connected by a jump, the discretization error clearly is concentrated in the element that contains the shock. With this understanding, a slight change is introduced. If the element that contains the shock is bounded by ξ_m and ξ_{m+1}, the error due to the mth element, i.e.,

$$\int_{\xi_m}^{\xi_{m+1}} R^2 \, \mathrm{d}\xi$$

in Eq. (3a)] is discarded for which $\xi_m \leq \xi \leq \xi_{m+1}$. As a consequence, the resulting algebraic system is decoupled into two subsystems. Such a decoupling reflects precisely the fact that the discontinuous solution with shock is actually a composite of two independent solutions of the first-order differential equation [Eq. (1)], one to satisfy the upstream condition V_1 and the other the downstream condition V_2. The shock jump condition is easily checked by extrapolating the upstream and downstream solutions into the mth element and "shock fitting," i.e., verifying the existence of a location ξ_{shock}, $\xi_m \leq \xi_{\text{shock}} \leq \xi_{m+1}$, where $[V^2] = 0$. The procedure thus involves also a systematic search for the shock element and the flexibility of the finite-element is of great advantage. A coarse mesh is enough to identify the possibility of a shock in the supersonic region, and a systematic local refinement of the mesh easily pinpoints the shock location. Further details concerning the implementation of the proposed method are given in the Appendix.

The extremum property of the least-square procedure is of added advantage to distinguish between the two numerical solutions, the continuous one and the one admitting a shock. With the same end conditions and the same mesh a comparison is made by evaluating the total least-square error,

$$\Upsilon(V) = \int_{-a}^{a} R^2 \mathrm{d}\xi \tag{4}$$

R being the "residual" already defined in Eq. (3b). The rarefaction shock is generally suppressed in the described procedure. However, using the special shock-element approach, the rarefaction shock can be obtained, if so desired.

Figure 2 shows the results of a typical example[2] for $\alpha = 0.1$ and $\gamma = 1.4$, which shows excellent agreement with the exact analytical solution. The two-dimensional problem using Sauer's[10] equations and an integral relations formulation similar to Holt's[11] has been treated as a quasi-one-dimensional problem by Wadia.[2] Due to the nature of the one-strip approximation the results are interpreted as defining a normal compression shock at the point where there is a jump in the average velocity. No attempt has yet been made to solve the two-strip formulation of the governing equations.

Concluding Remarks

The investigation of the model problem, corresponding to the one-dimensional transonic Laval nozzle, pinpoints the major source of difficulty of finding the weak shock by finite-element-type methods as the saddle-point nature of the family of possible solutions near the sonic point creates a very delicate situation. The large discretization error in the shock element is generally sufficient to mislead the computer program into picking out a neighboring continuous solution as the best compromise. A small supersonic pocket may appear in the approximate solution but is only illusory. The outlined search-and-fit procedure is quite effective to single out the correct solution with shock for the model problem. The method of integral relations in conjunction with the finite-element procedure may be used for further work on two-dimensional flows in a multistrip formulation. However, to obtain oblique shocks, it may become necessary to allow the velocity components to be discontinuous across the strips in the shock element, if the method of integral relations is to be used effectively.

Appendix

The shock is initially assumed to be in a certain element and the total error in that element

$$\int_{\xi_m}^{\xi_{m+1}} R^2 \mathrm{d}\xi$$

is not considered in the discretization process. If the shock is assumed in element 5 (Fig. 1), for instance, the error in that element

$$\int_{\xi_5}^{\xi_6} R^2 \, \mathrm{d}\xi$$

is neglected and the system of nonlinear, algebraic equations is thus obtained in a similar manner to that of Eq. (3a) and solved by Newton's method. Similarly, this procedure is repeated successively for all of the elements. With each set of results, the missing solution in the shock element is obtained by extrapolation.

The initial approximation to start the iterative scheme lies within a small range, V ranging within a band of width $2V(a)$. Using the conventional least-square method (without shock fitting) and an initial approximation of the values of the unknowns as a linear interpolation between the two specified boundary conditions a solution resembling the subsonic solution (Fig. 1) is determined. Now, if (for example) the shock is assumed to be in element 4, the initial approximation is taken (from the solution representing the subsonic solution, Fig. 1) as V_2, V_3, $-V_4$, V_5, and V_6; V_1 and V_7 being the boundary conditions. If the shock is considered in element 5, the initial approximation is V_2, V_3, $-V_4$, $-V_5$, and V_6. Hence, the program first determines the solution of the full equations (no shock fitting) and then automatically begins a search for the shock element systematically for the decoupled system of equations making use of the previously computed solution. Thus, the physics of the problem helps tremendously in using reasonably accurate initial estimates consequently saving machine time as the procedure searches for the shock.

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On Unsteady Transonic Shock Motions

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Introduction

IN recent investigations[1,2] into unsteady transonic flow theories, the problem of estimating the behavior of the oscillating shock wave in some analytic or semianalytic way has arisen. In particular, the variation of the shock oscillation magnitude and phase angle with reduced frequency is of interest. In the paper by Ashley,[1] the results of Seebass et al.[3] and Williams[2] are used to estimate approximate transonic flutter characteristics. The work of Seebass et al.[3] is based on a numerical shock-fitted solution of the linearized unsteady transonic small-disturbance equation. The theory of Williams[2] is based on the linear unsteady potential equation with the addition of an approximate treatment of the shock motion. At present, therefore, the choice seems to be between a numerical evaluation of the shock motion characteristics and the use of an approximate theory that may not entirely represent the correct shock movement. However, in a recent study[4] of the transonic indical method, Nixon derived an equation for the shock motion using the basic ideas of the indical method.[5] Apart from the assumption of linear perturbations of the shock location, the equation is general. It is the purpose of this Note to investigate the implications of this indicial equation with a view to establishing some pertinent facts of transonic shock motions in two-dimensional flow. A general theory of transonic shock motions is derived and the asymptotic behavior of the shock motion magnitude and phase angle for the high-frequency limit are derived. A simple approximate theory is also presented, the results of which are compared with the numerical predictions of Seebass et al.[3]

Analysis

The basic equation for the shock movement, denoted by $\delta X_s(t)$, derived in Ref. 4 [Eq. (11) of Ref. 4] is:

$$\delta X_s(t) = \delta X_{s_\epsilon}(t)\epsilon(0) + \int_0^t \delta X_{s_\epsilon}(\tau)\frac{\mathrm{d}\epsilon(t-\tau)}{\mathrm{d}t}\mathrm{d}\tau \tag{1}$$

where t is time and $\epsilon(t)$ is a time-dependent parameter typifying the unsteady motion. $\delta X_{s_\epsilon}(t)$ is the indicial response defined as the response of the shock location due to the instantaneous unit change in the parameter $\epsilon(t)$. A typical form of the indicial response for the shock location is shown in Fig. 1. This behavior was deduced from unpublished results of the numerical procedure of Ballhaus and Goorjian.[6]

Let the unsteady motion be simple harmonic motion, with the parameter $\epsilon(t)$ given by the equation

$$\epsilon(t) = \epsilon_0 + \epsilon_1 e^{i\omega t} \tag{2}$$

where ω is the frequency of the motion.

Define the term $\Delta X_{s_\epsilon}(t)$ by the following equation:

$$\Delta X_{s_\epsilon}(t) = \delta X_{s_\epsilon}(\infty) - \delta X_{s_\epsilon}(t) \tag{3}$$

where $\delta X_{s_\epsilon}(\infty)$ is the steady-state change in shock location due to a unit change in ϵ. Only the periodic motion is of interest; that is, the limit as $t \to \infty$. Then, substitution of Eqs. (2) and (3) into Eq. (1) gives:

$$\delta X_s(t) = \delta X_{s_\epsilon}(\infty)\epsilon_0 + \epsilon_1 e^{i\omega t}\left[\delta X_{s_\epsilon}(\infty) - i\omega\int_0^\infty \Delta X_{s_\epsilon}(\tau)e^{-i\omega t}\mathrm{d}\tau\right] \tag{4}$$

In order to investigate the shock motion further, some representation of the indicial response, $\delta X_{s_\epsilon}(t)$, is required. Bearing in mind the qualitative behavior of the shock indicial response given in Fig. 1, a suitable equation for the indicial response is:

$$\delta X_{s_\epsilon}(t) = \frac{\delta X_s(\infty)}{\epsilon_1}(1-e^{-at})\left(1+\sum_{n=1}^{N} b_n e^{-ant}\right) \tag{5}$$

where $\delta X_s(\infty)$ is the steady-state shock motion due to a change in ϵ of ϵ_1, a is some constant, and b_n are constant

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